Stabilities and Existence of Kähler-Einstein metrics

Xiaokui Yang

2006.12

Projective spaces

$\mathbb{P}^n$ is a standard model of projective manifolds.

- $\mathbb{P}^n$ is a compact Kähler manifold .

Projective spaces

$\mathbb{P}^n$ is a standard model of projective manifolds.

- $\mathbb{P}^n$ is a compact Kähler manifold .
- $(\mathbb{P}^n, \omega_{FS})$ is Kähler-Einstein manifold with constant Ricci curvature $n+1$,

$$Ric(\omega_{FS}) = (n+1)\omega_{FS}$$

Projective spaces

$\mathbb{P}^n$ is a standard model of projective manifolds.

- $\mathbb{P}^n$ is a compact Kähler manifold .
- $(\mathbb{P}^n, \omega_{FS})$ is Kähler-Einstein manifold with constant Ricci curvature $n+1$,

$$Ric(\omega_{FS}) = (n+1)\omega_{FS}$$

- $\mathbb{P}^n$ has positive constant scalar

$$\sigma = n(n+1)$$

Projective spaces

$\mathbb{P}^n$ is a standard model of projective manifolds.

- $\mathbb{P}^n$ is a compact Kähler manifold .
- $(\mathbb{P}^n, \omega_{FS})$ is Kähler-Einstein manifold with constant Ricci curvature $n+1$,

$$Ric(\omega_{FS}) = (n+1)\omega_{FS}$$

- $\mathbb{P}^n$ has positive constant scalar

$$\sigma = n(n+1)$$

- $\mathbb{P}^n$ has an ample(positive) line bundle– hyperplane bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ and ample(positive) tangent bundle;

Projective spaces

$\mathbb{P}^n$ is a standard model of projective manifolds.

- $\mathbb{P}^n$ is a compact Kähler manifold .
- $(\mathbb{P}^n, \omega_{FS})$ is Kähler-Einstein manifold with constant Ricci curvature $n+1$,

$$Ric(\omega_{FS}) = (n+1)\omega_{FS}$$

- $\mathbb{P}^n$ has positive constant scalar

$$\sigma = n(n+1)$$

- $\mathbb{P}^n$ has an ample(positive) line bundle– hyperplane bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ and ample(positive) tangent bundle;
- $T\mathbb{P}^n$ is stable and simple;

Projective spaces

$\mathbb{P}^n$ is a standard model of projective manifolds.

- $\mathbb{P}^n$ is a compact Kähler manifold .
- $(\mathbb{P}^n, \omega_{FS})$ is Kähler-Einstein manifold with constant Ricci curvature $n+1$,

$$Ric(\omega_{FS}) = (n+1)\omega_{FS}$$

- $\mathbb{P}^n$ has positive constant scalar

$$\sigma = n(n+1)$$

- $\mathbb{P}^n$ has an ample(positive) line bundle– hyperplane bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ and ample(positive) tangent bundle;
- $T\mathbb{P}^n$ is stable and simple;
- $\mathbb{P}^n$ has positive first Chern class;

Projective spaces

$\mathbb{P}^n$ is a standard model of projective manifolds.

- $\mathbb{P}^n$ is a compact Kähler manifold .
- $(\mathbb{P}^n, \omega_{FS})$ is Kähler-Einstein manifold with constant Ricci curvature $n+1$,

$$Ric(\omega_{FS}) = (n+1)\omega_{FS}$$

- $\mathbb{P}^n$ has positive constant scalar

$$\sigma = n(n+1)$$

- $\mathbb{P}^n$ has an ample(positive) line bundle– hyperplane bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ and ample(positive) tangent bundle;
- $T\mathbb{P}^n$ is stable and simple;
- $\mathbb{P}^n$ has positive first Chern class;
- ....

Some conjectures and theorems

(1) Frankel conjecture: $\mathbb{P}^n$ has positive tangent bundle $\Longrightarrow$ manifolds with positive tangent bundle is $\mathbb{P}^n$?

Some conjectures and theorems

(1) Frankel conjecture: $\mathbb{P}^n$ has positive tangent bundle $\Longrightarrow$ manifolds with positive tangent bundle is $\mathbb{P}^n$?

- **Theorem (Siu-Yau-Mori)**
  If M is a compact complex manifold with positive holomorphic bisectional curvature(ample tangent bundle), then M is biholomorphic to some projective space.

Some conjectures and theorems

(1) Frankel conjecture: $\mathbb{P}^n$ has positive tangent bundle $\Longrightarrow$ manifolds with positive tangent bundle is $\mathbb{P}^n$?

- **Theorem (Siu-Yau-Mori)**
  If M is a compact complex manifold with positive holomorphic bisectional curvature(ample tangent bundle), then M is biholomorphic to some projective space.
- Griffiths conjecture: positive $\Longleftrightarrow$ ample.

Some conjectures and theorems

(2) "Conjecture": $c_1 > 0 \Longrightarrow \exists$ Kähler-Einstein metric or constant scalar curvature metric?

Some conjectures and theorems

(2) "Conjecture": $c_1 > 0 \Longrightarrow \exists$ Kähler-Einstein metric or constant scalar curvature metric?

- No, there exists manifold with $c_1 > 0$ which admits no Kähler-Einstein metric.

Some conjectures and theorems

(2) "Conjecture": $c_1 > 0 \Longrightarrow \; \exists$ Kähler-Einstein metric or constant scalar curvature metric?

- No, there exists manifold with $c_1 > 0$ which admits no Kähler-Einstein metric.
- There are some obstructions to the existence of Kähler-Einstein metrics, such as Calabi-Futaki invariants.

Some conjectures and theorems

(3) Yau's conjecture: the existence of the canonical metrics are equivalent to "certain stabilities" of manifolds.

Some conjectures and theorems

(3) Yau's conjecture: the existence of the canonical metrics are equivalent to "certain stabilities" of manifolds.

- The most important case is the manifolds with positive first Chern classes, i.e. manifolds with ample anti-canonical line bundles.

Positivity in complex geometry

Siu introduced the following definition:

Positivity in complex geometry

Siu introduced the following definition:

If (M, h) is a Kähler manifold. The Riemannian curvature tensor $R_{\alpha\overline{\beta}\gamma\overline{\delta}}$ is said to be strongly positive if

$$\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\overline{\beta}\gamma\overline{\delta}}(A^\alpha\overline{B}^\beta - C^\alpha\overline{D}^\beta)(\overline{A^\delta\overline{B}^\gamma - C^\delta\overline{D}^\gamma}) > 0$$

for any nonzero complex matrix $(A^\alpha\overline{B}^\beta - C^\alpha\overline{D}^\beta)$;

Positivity in complex geometry

Siu introduced the following definition:

If (M, h) is a Kähler manifold. The Riemannian curvature tensor $R_{\alpha\overline{\beta}\gamma\overline{\delta}}$ is said to be strongly positive if

$$\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\overline{\beta}\gamma\overline{\delta}}(A^\alpha\overline{B}^\beta - C^\alpha\overline{D}^\beta)(\overline{A^\delta\overline{B}^\gamma - C^\delta\overline{D}^\gamma}) > 0$$

for any nonzero complex matrix $(A^\alpha\overline{B}^\beta - C^\alpha\overline{D}^\beta)$;

The Riemannian curvature tensor $R_{\alpha\overline{\beta}\gamma\overline{\delta}}$ is said to be very strongly positive if

$$\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\overline{\beta}\gamma\overline{\delta}} u^{\alpha\delta}\overline{u}^{\beta\gamma} > 0$$

for any nonzero complex matrix $(u^{\alpha\beta})$, where

$$R_{\alpha\overline{\beta}\gamma\overline{\delta}} = -\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial w^\gamma \partial \overline{w}^\delta} + h^{\lambda\overline{\mu}} \frac{\partial h_{\alpha\overline{\mu}}}{\partial w^\gamma} \frac{h_{\lambda\overline{\beta}}}{\partial \overline{w}^\delta}$$

in the local coordinate (w^α).

Positivity in complex geometry

Positivity in complex geometry

Theorem

Let $f : (N, \omega_N) \longrightarrow (M, \omega_M)$ be a harmonic map between a compact hermitian manifold (N, ω_N) and a Kähler manifold (M, ω_M). If the hermitian metric satisfies $\partial\overline{\partial}\omega_N = 0$ and the curvature tensor of M is strongly semi-negative and strongly negative at $f(P)$ for some point $P \in N$ with $\mathrm{rank}_{\mathbb{R}} df \geq 4$ at P, then f is holomorphic or anti-holomorphic.

Positivity in complex geometry

Theorem

Let $f : (N, \omega_N) \longrightarrow (M, \omega_M)$ be a harmonic map between a compact hermitian manifold (N, ω_N) and a Kähler manifold (M, ω_M). If the hermitian metric satisfies $\partial\overline{\partial}\omega_N = 0$ and the curvature tensor of M is strongly semi-negative and strongly negative at $f(P)$ for some point $P \in N$ with $\mathrm{rank}_{\mathbb{R}} df \geq 4$ at P, then f is holomorphic or anti-holomorphic.

This is the generalization of Siu's result:

Theorem

Let $f : (N, \omega_N) \longrightarrow (M, \omega_M)$ be a harmonic map between compact Kähler manifolds. If the curvature tensor of M is strongly semi-negative and strongly negative at $f(P)$ for some point $P \in N$ with $\mathrm{rank}_{\mathbb{R}} df \geq 4$ at P, then f is holomorphic or anti-holomorphic.

Manifolds with definite first Chern classes

- If (M, ω) is Kähler-Einstein, then M has definite first Chern class.

Manifolds with definite first Chern classes

- If (M, ω) is Kähler-Einstein, then M has definite first Chern class.
- Conversely, if M has definite first Chern class, does M admit Kähler-Einstein metric?

Manifolds with definite first Chern classes

- If (M, ω) is Kähler-Einstein, then M has definite first Chern class.
- Conversely, if M has definite first Chern class, does M admit Kähler-Einstein metric?
- This question is equivalent to the following Monge-Ampere equation:

$$\begin{cases} \frac{\det(g_{i\bar{j}}+\varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^{-k\varphi+F}, \quad (g_{i\bar{j}}+\varphi_{i\bar{j}}) > 0 \\ \int_M e^{-k\varphi+F} dV_g = vol(M,g) \end{cases} \tag{2.1}$$

Yau's solution of Calabi conjecture

Yau's solution of Calabi conjecture

- **Theorem (Aubin &Yau)**
  Let M be any compactKähler manifold with $c_1(M) < 0$, then there exists a unique K-E metric ω with $Ric(\omega) = -\omega$.

Yau's solution of Calabi conjecture

- Theorem (Aubin &Yau)

  Let M be any compactKähler manifold with $c_1(M) < 0$, then there exists a unique K-E metric ω with $Ric(\omega) = -\omega$.

- Theorem (Calabi-Yau)

  Let (M, ω) be a compact Kähler manifold. Given any real closed (1,1) form

$$\widetilde{\rho} = \frac{\sqrt{-1}}{2\pi} \widetilde{R}_{ij} dz^i \wedge d\overline{z}^j$$

  which represents $c_1(M) = [\widetilde{\rho}]$, then there exists a unique Kähler form $\widetilde{\omega} \in [\omega]$ such that the corresponding Ricci form $Ric(\widetilde{\omega}) = \widetilde{\rho}$.

Some corollaries of Calabi-Yau's theorem

Some corollaries of Calabi-Yau's theorem

- Corollary
  If M is a Kähler manifold with $c_1(M) = 0$, then there exists a Ricci flat Kähler metric ω.

Some corollaries of Calabi-Yau's theorem

- **Corollary**

  If M is a Kähler manifold with $c_1(M) = 0$, then there exists a Ricci flat Kähler metric ω.

- **Corollary (Kobayashi-Yau)**

  If $c_1(M) > 0$, then M has a Kähler metric with positive Ricci curvature, and so is simply connected.

Definition

(1) A holomorphic vector bundle E over a compact complex manifold (M, ω) is simple if

$$\dim_{\mathbb{C}} H^0(M, Hom_{\mathbb{C}}(E, E)) = 1$$

Definition

(1) A holomorphic vector bundle E over a compact complex manifold (M, ω) is simple if

$$\dim_{\mathbb{C}} H^0(M, Hom_{\mathbb{C}}(E, E)) = 1$$

(2) If (E, h) is a hermitian vector bundle over a compact complex manifold (M, ω), then E is called weakly Hermitian-Einstein if

$$\Lambda_\omega \Theta^E = \varphi I$$

where Θ^E is the Chern curvature of (E, h) and φ is a global defined function.

Definition

(1) A holomorphic vector bundle E over a compact complex manifold (M, ω) is simple if

$$\dim_{\mathbb{C}} H^0(M, Hom_{\mathbb{C}}(E, E)) = 1$$

(2) If (E, h) is a hermitian vector bundle over a compact complex manifold (M, ω), then E is called weakly Hermitian-Einstein if

$$\Lambda_\omega \Theta^E = \varphi I$$

where Θ^E is the Chern curvature of (E, h) and φ is a global defined function.

- Moreover, if φ is a global constant, then (E, h) is called a Hermitian-Einstein. The corresponding Chern connection is called the Hermitian-Yang-Mills connection

(3) A holomorphic vector bundle E over a compact Kähler manifold (M, ω) is called $[\omega]$-stable(resp. $[\omega]$-semi-stable) if for any subbundle F of E with $0 < rk(F) < rk(E)$, we have

$$\mu(F) < \mu(E)(\quad \text{resp.} \quad \mu(F) \leq \mu(E))$$

where

$$\mu(E) = \frac{\int_M c_1(E) \wedge \omega^{n-1}}{rk(E)}$$

(3) A holomorphic vector bundle E over a compact Kähler manifold (M, ω) is called $[\omega]$-stable(resp. $[\omega]$-semi-stable) if for any subbundle F of E with $0 < rk(F) < rk(E)$, we have

$$\mu(F) < \mu(E)(\quad \text{resp.} \quad \mu(F) \leq \mu(E))$$

where

$$\mu(E) = \frac{\int_M c_1(E) \wedge \omega^{n-1}}{rk(E)}$$

(4) If $E = E_1 \oplus \cdots \oplus E_s$ and E_i are stable and

$$\mu(E_1) = \cdots = \mu(E_s)$$

then E is called polystable.

Relations

- weakly Hermitian-Einstein $\Longrightarrow$ Hermitian-Einstein;

Relations

- weakly Hermitian-Einstein $\Longrightarrow$ Hermitian-Einstein;
- stable $\Longrightarrow$ simple $\Longrightarrow$ irreducible;

Relations

- weakly Hermitian-Einstein $\Longrightarrow$ Hermitian-Einstein;
- stable $\Longrightarrow$ simple $\Longrightarrow$ irreducible;
- polystable $\Longrightarrow$ semi-stable.

Theorem (Lubke-Kobayashi)

(1) If (E, h) is a Hermitian-Einstein vector bundle over a compact Kähler manifold (M, ω). Then E is polystable, i.e.(E, h) is a direct sum

$$(E, h) = (E_1, h_1) \bigoplus \cdots \bigoplus (E_k, h_k)$$

of $[\omega]$-stable Hermitian-Einstein vector bundles $(E_1, h_1), \cdots, (E_s, h_s)$ with

$$\mu(E) = \mu(E_1) = \cdots = \mu(E_s)$$

Theorem (Lubke-Kobayashi)

(1) *If (E, h) is a Hermitian-Einstein vector bundle over a compact Kähler manifold (M, ω). Then E is polystable, i.e.(E, h) is a direct sum*

$$(E, h) = (E_1, h_1) \bigoplus \cdots \bigoplus (E_k, h_k)$$

of $[\omega]$-stable Hermitian-Einstein vector bundles $(E_1, h_1), \cdots, (E_s, h_s)$ with

$$\mu(E) = \mu(E_1) = \cdots = \mu(E_s)$$

(2) *Let (E, h) be a Hermitian-Einstein vector vector bundle over a compact Kähler manifold (M, ω). If E is simple, then E is $[\omega]$-stable.*

Theorem (Donaldson-Ulenbeck-Yau)

If E is a stable vector bundle over a compact Kähler manifold (M, ω), then E admits a unique Hermitian-Einstein metric.

Theorem (Donaldson-Ulenbeck-Yau)

If E is a stable vector bundle over a compact Kähler manifold (M, ω), then E admits a unique Hermitian-Einstein metric.

Theorem (HKDUY Correspondence)

If (E, h) is a hermitian vector bundle over a compact Kähler manifold (M, ω), then (E, h) is polystable if and only if it admits Hermitian-Einstein metrics (Hermitian-Yang-Mills connections).

Yau's conjecture

- Yau conjectured that the existence of Kähler-Einstein (cscK) metric is equivalence to certain stabilities of manifolds.

Yau's conjecture

- Yau conjectured that the existence of Kähler-Einstein (cscK) metric is equivalence to certain stabilities of manifolds.
- We could not use the "stabilities" of the vector bundle for the metrics on E and on TM are different.

Stability in the sense of Gieseker

Stability in the sense of Gieseker

Definition

If (M, L) is a polarized manifold, and E is a holomorphic vector bundle over M. E is called Gieseker stable(resp. semi-stable) if for every subbundle $F \subset E$ with $0 < rk(F) < rk(E)$, the inequality

$$\frac{\chi(F \otimes L^k)}{rk(F)} < \frac{\chi(E \otimes L^k)}{rk(E)} \left(\quad \text{resp.} \quad \frac{\chi(F \otimes L^k)}{rk(F)} \leq \frac{\chi(E \otimes L^k)}{rk(E)} \right)$$

holds for sufficient large integers k.

Stability in the sense of Gieseker

By the Riemann-Roch theorem, we have

$$\chi(E \otimes L^k) = \int_M ch(E \otimes L^k) \cdot Td(M)$$

Stability in the sense of Gieseker

By the Riemann-Roch theorem, we have

$$\chi(E \otimes L^k) = \int_M ch(E \otimes L^k) \cdot Td(M)$$

$$\chi(E \otimes L^k) = \int_M \frac{rk^n}{n!} c_1^n(L) + \frac{k^{n-1}}{(n-1)!} c_1^{n-1}(L)\big(c_1(E) + \frac{r}{2} c_1(M)\big) + O(k^{n-2})$$

Stability in the sense of Gieseker

By the Riemann-Roch theorem, we have

$$\chi(E \otimes L^k) = \int_M ch(E \otimes L^k) \cdot Td(M)$$

$$\chi(E \otimes L^k) = \int_M \frac{rk^n}{n!} c_1^n(L) + \frac{k^{n-1}}{(n-1)!} c_1^{n-1}(L)\big(c_1(E) + \frac{r}{2} c_1(M)\big) + O(k^{n-2})$$

Then

$$\frac{\chi(E \otimes L^k)}{rk(E)} - \frac{\chi(F \otimes L^k)}{rk(F)} = \frac{k^{n-1}}{(n-1)!}(\mu(E) - \mu(F)) + O(k^{n-2})$$

Stability in the sense of Gieseker

By the Riemann-Roch theorem, we have

$$\chi(E \otimes L^k) = \int_M ch(E \otimes L^k) \cdot Td(M)$$

$$\chi(E \otimes L^k) = \int_M \frac{rk^n}{n!} c_1^n(L) + \frac{k^{n-1}}{(n-1)!} c_1^{n-1}(L)\big(c_1(E) + \frac{r}{2} c_1(M)\big) + O(k^{n-2})$$

Then

$$\frac{\chi(E \otimes L^k)}{rk(E)} - \frac{\chi(F \otimes L^k)}{rk(F)} = \frac{k^{n-1}}{(n-1)!} (\mu(E) - \mu(F)) + O(k^{n-2})$$

Stable $\Longrightarrow$ Gieseker stable $\Longrightarrow$ Gieseker semistable $\Longrightarrow$ semistable

GIT quotient

If V is a family of objects, and there is an equivalence relation $\sim$ on V, what is $V/\sim$?

GIT quotient

If V is a family of objects, and there is an equivalence relation $\sim$ on V, what is $V/\sim$?

If G is a group, what is the quotient V/G of the group action

$$G \times V \longrightarrow V?$$

GIT quotient

If V is a family of objects, and there is an equivalence relation $\sim$ on V, what is $V/\sim$?

If G is a group, what is the quotient V/G of the group action

$$G \times V \longrightarrow V?$$

In general, the quotients are "bad"!

Example of "bad" action

Given an integer N, let

$$V_N = \left\{ \sum_{i+j=N} a_{ij} X^i Y^j \right\} \subset \mathbb{C}[X, Y]$$

and

$$\mathbb{P}(V_N) = V_N / \mathbb{C}^* = \mathbb{P}^N$$

The group $G = SL(2, \mathbb{C})$ acts on V_N by

$$V_N \times G \longrightarrow V_N, \quad (g, P) = P^g$$

$$P^g(X, Y) = P\left(g^{-1}\begin{pmatrix} X \\ Y \end{pmatrix}\right)$$

There is a natural induced action

$$\mathbb{P}^N \times G \longrightarrow \mathbb{P}^N$$

The action is so bad that there is no Hausdorff topology on $\mathbb{P}^N/G$.

GIT quotient

Let G be a reductive algebraic group, and V is a n-dimensional representation of G. That is a group action

$$\rho : G \times V \longrightarrow V$$

with $\rho_g : V \longrightarrow V$ a linear isomorphism map.

GIT quotient

Let G be a reductive algebraic group, and V is a n-dimensional representation of G. That is a group action

$$\rho : G \times V \longrightarrow V$$

with $\rho_g : V \longrightarrow V$ a linear isomorphism map.

Then for a point $x \in V$:

x unstable	x semi-stable	x stable
$0 \in \overline{O(x)}$	$0 \notin \overline{O(x)}$	(1). $O(x)$ is closed in V; (2). Stabilizer G_x is finite
$\forall f \in k[X_1, \cdots, X_n]^G$ with $\deg f \geq 1$, $f(x) = 0$	$\exists f \in k[X_1, \cdots, X_n]^G$ with $\deg f \geq 1$, such that $f(x) \neq 0$	(1). $\dim O(x) = \dim G$; (2). $\exists f \in k[X_1, \cdots, X_n]^G$ with $\deg f \geq 1$, such that $f(x) \neq 0$; (3). the action of G on X_f is closed.
$\exists$ 1-PS λ of G such that $\mu(x, \lambda) < 0$	$\forall$ 1-PS λ of G such that $\mu(x, \lambda) \geq 0$	$\forall$ 1-PS λ of G such that $\mu(x, \lambda) > 0$

Hilbert Scheme-Moduli of polarized manifolds

Let

$$A = \{(X, L) \mid L \quad \text{is very ample} \quad \}$$

be the set of the objects. The equivalent relations on A is given by

$$(X, L) \sim (X', L') \Longleftrightarrow \exists \quad \text{isomorphism} \quad \tau : X \longrightarrow X' \quad \text{such that} \quad \tau^* L' \cong L$$

Hilbert Scheme-Moduli of polarized manifolds

Let

$$A = \{(X, L) \mid L \quad \text{is very ample} \quad \}$$

be the set of the objects. The equivalent relations on A is given by

$$(X, L) \sim (X', L') \Longleftrightarrow \exists \quad \text{isomorphism} \quad \tau : X \longrightarrow X' \quad \text{such that} \quad \tau^* L' \cong L$$

If (X, L) is a polarized manifold, then we know the Euler-Poincare characteristic

$$P(k) = \chi(X, L^k)$$

is a polynomial of degree $n(= \dim X)$ in k. Now fix a polynomial $P(T) \in \mathbb{Q}(T)$ with degree n, then we can consider the moduli problem for

$$A_P = \{(X, L) \mid L \quad \text{is very ample with Hilbert polynomial } P \quad \}$$

and the equivalent relations is the same as A.

Hilbert Scheme-Moduli of polarized manifolds

Let

$$A = \{(X, L) \mid L \quad \text{is very ample} \quad \}$$

be the set of the objects. The equivalent relations on A is given by

$$(X, L) \sim (X', L') \Longleftrightarrow \exists \quad \text{isomorphism} \quad \tau : X \longrightarrow X' \quad \text{such that} \quad \tau^* L' \cong L$$

If (X, L) is a polarized manifold, then we know the Euler-Poincare characteristic

$$P(k) = \chi(X, L^k)$$

is a polynomial of degree $n(= \dim X)$ in k. Now fix a polynomial $P(T) \in \mathbb{Q}(T)$ with degree n, then we can consider the moduli problem for

$$A_P = \{(X, L) \mid L \quad \text{is very ample with Hilbert polynomial } P \quad \}$$

and the equivalent relations is the same as A.

What is $A_P / \sim$?

Hilbert Scheme-Moduli of polarized manifolds

- Now we set $\mathbb{P}^N = \mathbb{P}^{P(1)-1}$. Consider the consider the contravariant functor

$$\mathcal{H}ilb^P_{\mathbb{P}^N} : \{ \quad \text{schemes} \quad \} \longrightarrow \{ \quad \text{sets} \quad \}$$

which is given by

$$\mathcal{H}ilb^P_{\mathbb{P}^N}(U) = \left\{ Z \subset \mathbb{P}^N \times U : \begin{array}{c} Z \quad \text{is a closed subscheme} \\ \begin{array}{ccc} Z & \xrightarrow{i} & \mathbb{P}^N \times U \\ \pi \downarrow & & \downarrow p_2 \\ U & \xrightarrow{=} & U \end{array} \end{array} \quad \pi \quad \text{is flat and } Z \text{ with Hilbert polynomial } P. \right\}$$

where " with Hilbert polynomial P" means: for $u \in U$, the Hilbert polynomial in u is

$$P(k) = \chi(Z_u, i^*_{Z_u}(\mathcal{O}_{\mathbb{P}^N}(k))), \qquad i_{Z_u} : Z_u \hookrightarrow \mathbb{P}^N$$

In fact, we know

$$(Z_u, i^*_{Z_u}(\mathcal{O}_{\mathbb{P}^N}(1))) \in A_P$$

Hilbert Scheme-Moduli of polarized manifolds

The following theorem is proved by Grothendieck with simplifications by Mumford.

Hilbert Scheme-Moduli of polarized manifolds

The following theorem is proved by Grothendieck with simplifications by Mumford.

Theorem (Grothendieck)

The functor $\mathcal{H}ilb^P_{\mathbb{P}^N}$ *is represented by a projective scheme* $(Hilb^P_{\mathbb{P}^N}, \Phi)$ *with the universal family* $Univ^P_{\mathbb{P}^N}$ *over* $Hilb^P_{\mathbb{P}^N}$

$$\begin{array}{ccc} Univ^P_{\mathbb{P}^N} & \xrightarrow{\subset} & \mathbb{P}^N \times Hilb^P_{\mathbb{P}^N} \\ \downarrow{\scriptstyle\pi} & & \\ Hilb^P_{\mathbb{P}^N} & & \end{array}$$

i.e.

$$Hilb^P_{\mathbb{P}^N} \cong A_p/\sim$$

Hilbert-Mumford stability of polarized manifolds

If (X, L) is a polarized manifold with Hilbert polynomial h. Then there exists a number r_0 such that if $r \geq r_0$, then L^r is very ample.

Hilbert-Mumford stability of polarized manifolds

If (X, L) is a polarized manifold with Hilbert polynomial h. Then there exists a number r_0 such that if $r \geq r_0$, then L^r is very ample.

The Kodaira embedding

$$\Phi_k : X \longrightarrow \mathbb{P}^{h(r)-1}$$

gives an exact sequence over $\mathbb{P}^{h(r)-1}$

$$0 \longrightarrow \mathfrak{I}_X(k) \longrightarrow \mathcal{O}_{\mathbb{P}^{h(r)-1}}(k) \longrightarrow \Phi_{k*}(\mathcal{O}_X)(k) \longrightarrow 0$$

Hilbert-Mumford stability of polarized manifolds

For large k we get

$$0 \longrightarrow H^0(\mathbb{P}^{h(r)-1}, \mathfrak{I}_X(k)) \longrightarrow S^k\mathbb{C}^{h(r)*} \cong S^k(H^0(X, L^r)) \longrightarrow H^0(X, L^{kr}) \longrightarrow 0$$

Hilbert-Mumford stability of polarized manifolds

For large k we get

$$0 \longrightarrow H^0(\mathbb{P}^{h(r)-1}, \mathfrak{I}_X(k)) \longrightarrow S^k\mathbb{C}^{h(r)*} \cong S^k(H^0(X, L^r)) \longrightarrow H^0(X, L^{kr}) \longrightarrow 0$$

- The k-th Hilbert point of (X, L^r) is determined in the Grassmannian

$$x_{r,k} \in \mathbb{G} = Grass\left(h(rk), \binom{h(r)+k-1}{k}\right) \hookrightarrow \mathbb{P}^{\binom{\binom{h(r)+k-1}{k}}{h(rk)}-1}$$

We know that $SL(h(r), \mathbb{C})$ acts linearly on $\mathbb{P}^{h(r)-1} = \mathbb{C}^{h(r)}/\mathbb{C}^*$ and so on $\mathbb{G}$.

Hilbert-Mumford stability of polarized manifolds

For large k we get

$$0 \longrightarrow H^0(\mathbb{P}^{h(r)-1}, \mathfrak{I}_X(k)) \longrightarrow S^k\mathbb{C}^{h(r)*} \cong S^k(H^0(X, L^r)) \longrightarrow H^0(X, L^{kr}) \longrightarrow 0$$

- The k-th Hilbert point of (X, L^r) is determined in the Grassmannian

$$x_{r,k} \in \mathbb{G} = Grass\left(h(rk), \binom{h(r)+k-1}{k}\right) \hookrightarrow \mathbb{P}^{\binom{\binom{h(r)+k-1}{k}}{h(rk)}-1}$$

We know that $SL(h(r), \mathbb{C})$ acts linearly on $\mathbb{P}^{h(r)-1} = \mathbb{C}^{h(r)}/\mathbb{C}^*$ and so on $\mathbb{G}$.
We can defined the Hilbert-Mumford stability of (X, L) by the Hilbert point $x_{r,k}$.

Chow-Mumford stability of polarized manifolds

Now we define a set of objects for the given positive integers N, n, d

$$A_{N,n,d} = \{X \subset \mathbb{P}^N \mid \dim_{\mathbb{C}} X = n, \deg(X) = d\}$$

and the equivalence relations are defined by

$$X \sim X' \Longleftrightarrow, \exists g \in SL(N+1, \mathbb{C}), \ g^*(X) = X'$$

Chow-Mumford stability of polarized manifolds

Now we define a set of objects for the given positive integers N, n, d

$$A_{N,n,d} = \{X \subset \mathbb{P}^N \mid \dim_{\mathbb{C}} X = n, \deg(X) = d\}$$

and the equivalence relations are defined by

$$X \sim X' \Longleftrightarrow, \exists g \in SL(N+1, \mathbb{C}), \ g^*(X) = X'$$

Then we can define the Chow functor

$$\mathcal{C}how_{N,n,d} : \{ \quad \text{varieties} \quad \} \longrightarrow \{ \quad \text{sets} \quad \}$$

by

$$\mathcal{C}how_{N,n,d}(U) = \{ \quad \text{families in } A_{N,n,d} \text{ parameterized by } U \quad \}$$

Chow-Mumford stability of polarized manifolds

Now we define a set of objects for the given positive integers N, n, d

$$A_{N,n,d} = \{X \subset \mathbb{P}^N \mid \dim_{\mathbb{C}} X = n, \deg(X) = d\}$$

and the equivalence relations are defined by

$$X \sim X' \Longleftrightarrow, \exists g \in SL(N+1, \mathbb{C}),\ g^*(X) = X'$$

Then we can define the Chow functor

$$\mathcal{C}how_{N,n,d} : \{ \quad \text{varieties} \quad \} \longrightarrow \{ \quad \text{sets} \quad \}$$

by

$$\mathcal{C}how_{N,n,d}(U) = \{ \quad \text{families in } A_{N,n,d} \text{ parameterized by } U \quad \}$$

Theorem

The functor $\mathcal{C}how_{N,n,d}$ is represented by a variety $Chow_{N,n,d}$.

Chow-Mumford stability of polarized manifolds

We have

$$\begin{array}{ccccc} Univ_{N,n,d} & \xrightarrow{\ i\ } & \mathbb{P}^N \times Chow_{N,n,d} & \xrightarrow{\ \pi_2\ } & \mathbb{P}^N \\ & \searrow^{\pi} & \downarrow & & \\ & & Chow_{N,n,d} & & \end{array}$$

Chow-Mumford stability of polarized manifolds

We have

$$\begin{array}{ccccc} Univ_{N,n,d} & \xrightarrow{\quad i \quad} & \mathbb{P}^N \times Chow_{N,n,d} & \xrightarrow{\quad \pi_2 \quad} & \mathbb{P}^N \\ & \searrow^{\pi} & \downarrow & & \\ & & Chow_{N,n,d} & & \end{array}$$

The algebraic group $G = SL(N+1, \mathbb{C})$ acts on $\mathbb{P}^N$ naturally, and consequently will acts on $Chow_{N,n,d}$ and $Univ_{N,n,d}$ equivariantly. Let ν_0 be a large positive constant, then there is a very ample line bundle

$$\mathcal{L} = \det(\pi_* i^* (\pi_2^* (\mathcal{O}_{\mathbb{P}^N}(\nu)))), \qquad \nu \geq \nu_0$$

on $Chow_{N,n,d}$ and also $\mathcal{L}$ is G-linearized.

Chow-Mumford stability of polarized manifolds

We have

$$\begin{array}{ccccc} Univ_{N,n,d} & \xrightarrow{\quad i \quad} & \mathbb{P}^N \times Chow_{N,n,d} & \xrightarrow{\quad \pi_2 \quad} & \mathbb{P}^N \\ & \searrow^{\pi} & \downarrow & & \\ & & Chow_{N,n,d} & & \end{array}$$

The algebraic group $G = SL(N+1, \mathbb{C})$ acts on $\mathbb{P}^N$ naturally, and consequently will acts on $Chow_{N,n,d}$ and $Univ_{N,n,d}$ equivariantly. Let ν_0 be a large positive constant, then there is a very ample line bundle

$$\mathcal{L} = \det(\pi_* i^* (\pi_2^* (\mathcal{O}_{\mathbb{P}^N}(\nu)))), \qquad \nu \geq \nu_0$$

on $Chow_{N,n,d}$ and also $\mathcal{L}$ is G-linearized.

For we have

$$Chow_{N,n,d} = A_{N,n,d} / \sim$$

Then if $X \in A_{N,n,d}$, we denote the equivalent class $[X] \in Chow_{N,n,d}$ by $Chow(X)$ called the Chow point of X.

Relations of stabilities

Chow-Mumford stable $\Longrightarrow$ Hilbert Mumford stable $\Longrightarrow$
Hilbert-Mumford semistable $\Longrightarrow$ Chow-Mumford semistable.

$$w(k) = Cek^{n+1} + O(k^n), \ \ C > 0$$

Asymptotic expansion of Bergman functions

If (L, h) is a positive line bundle over a compact Kähler manifold (X, ω) with $\omega = \sqrt{-1}\Theta^L$. Let $\Gamma(X, L^k)$ be the set a smooth sections. Then there is a natural inner product

$$(s_1, s_2) = \int_X \langle s_1(z), s_2(z)\rangle_h dV_z$$

for any $s_1, s_2 \in \Gamma(X, L^k)$ where

$$dV = \frac{\omega^n}{n!}$$

Asymptotic expansion of Bergman functions

If (L, h) is a positive line bundle over a compact Kähler manifold (X, ω) with $\omega = \sqrt{-1}\Theta^L$. Let $\Gamma(X, L^k)$ be the set a smooth sections. Then there is a natural inner product

$$(s_1, s_2) = \int_X \langle s_1(z), s_2(z)\rangle_h dV_z$$

for any $s_1, s_2 \in \Gamma(X, L^k)$ where

$$dV = \frac{\omega^n}{n!}$$

Then there is a natural orthogonal projection

$$P_k : \mathcal{L}^2(X, L^k) \longrightarrow H^0(X, L^k)$$

The Bergman kernel $P_k(z, z')$ is the smooth kernel of the projection

$$(P_k S)(z) = \int_X P_k(z, z')S(z')dV_{z'}$$

for any $S \in \mathcal{L}^2(X, L^k)$.

Asymptotic expansion of Bergman functions

We denote by $B_k(z) = P_k(z, z)$ the restriction on the diagonal. It is obvious that $B_k(z) \in End(L^k)_z$. So we can regard B_k as a smooth function on X.

Asymptotic expansion of Bergman functions

We denote by $B_k(z) = P_k(z, z)$ the restriction on the diagonal. It is obvious that $B_k(z) \in End(L^k)_z$. So we can regard B_k as a smooth function on X.

Let $s_0, \cdots, s_{N_k}$ be the orhonormal basis of $H^0(X, L^k)$ with respect to the $\mathcal{L}^2$ metric, i.e.

$$(s_\alpha, s_\beta)_{\mathcal{L}^2} = \int_X \langle s_\alpha(z), s_\beta(z)\rangle_h dV_z = \delta_{\alpha\beta}$$

Asymptotic expansion of Bergman functions

We denote by $B_k(z) = P_k(z, z)$ the restriction on the diagonal. It is obvious that $B_k(z) \in End(L^k)_z$. So we can regard B_k as a smooth function on X.

Let $s_0, \cdots, s_{N_k}$ be the orhonormal basis of $H^0(X, L^k)$ with respect to the $\mathcal{L}^2$ metric, i.e.

$$(s_\alpha, s_\beta)_{\mathcal{L}^2} = \int_X \langle s_\alpha(z), s_\beta(z)\rangle_h dV_z = \delta_{\alpha\beta}$$

-

Theorem

We have the relations

$$B_k(z) = \sum_{\alpha=0}^{N_k} |s_\alpha|_h^2(z)$$

where $|\clubsuit|_h$ *is the fiberwise norm with respect to the metric h. Moreover, it is independent on the choice of the base, and intrinsically we have*

$$B_k(z) = \sup\{|\varphi|_h^2(z) \mid \ \|\varphi\|_{\mathcal{L}^2} = 1\}$$

Asymptotic expansion of Bergman functions

Asymptotic expansion of Bergman functions

Theorem

If (L, h) is a positive line bundle over a compact Kähler manifold (X, ω) and $\omega = \sqrt{-1}\Theta_h^L$. The orthonormal basis $s_0, \cdots, s_{N_k} \in H^0(X, L^k)$ with respect to the $\mathcal{L}^2$ inner product

$$(s_i, s_j) = \int_X \langle s_i(x), s_j(x) \rangle_h dV$$

gives the Kodaira embedding

$$\Phi_k : X \longrightarrow \mathbb{P}(H^0(M, L^k)^*) \cong \mathbb{P}^{N_k}, \qquad x \longrightarrow [s_0(x), \cdots, s_{N_k}(x)]$$

satisfies

$$\frac{1}{k}\Phi_k^*(\omega_{FS}) - \omega = \frac{\sqrt{-1}}{k}\partial\overline{\partial}\log B_k$$

Asymptotic expansion of Bergman functions

Asymptotic expansion of Bergman functions

Corollary

$$\dim_{\mathbb{C}} H^0(X, L^k) = \int_X B_k(\omega) dV_\omega = a_0 k^n + a_1 k^{n-1} + a_2 k^{n-2} + \cdots + a_n$$

where

$$a_0 = Vol(X, \omega), a_1 = \frac{1}{2} \int_X s_\omega dV_\omega$$

Asymptotic expansion of Bergman functions

Corollary

$$\dim_{\mathbb{C}} H^0(X, L^k) = \int_X B_k(\omega) dV_\omega = a_0 k^n + a_1 k^{n-1} + a_2 k^{n-2} + \cdots + a_n$$

where

$$a_0 = Vol(X, \omega), a_1 = \frac{1}{2} \int_X s_\omega dV_\omega$$

Theorem

$$\frac{B_k(\omega, z)}{Vol(X, \omega)} \sim A_0(z) k^n + A_1(z) k^{n-1} + \cdots$$

where

$$A_0 = 1, A_1 = \frac{s_\omega}{2}$$

$$\|B_k(z) - \sum_{i=0}^{N} A_i(z) k^{n-i}\|_{C^r(X)} \leq K_{r,N} k^{n-N-1}$$

Balanced polarization

If L is an ample vector bundle over a compact complex manifold X. Let $B = \{s_0, s_1, \cdots, s_{N_k}\}$ be a basis of $V_k = H^0(X, L^k)$ such that it gives an embedding

$$\Phi_k : X \longrightarrow \mathbb{P}(V^*), \quad x \longrightarrow [s_0(x), \cdots, s_{N_k}(x)]$$

We choose the local frame e_L of L, then

$$s_\alpha(x) = f_\alpha(x) e_L^{\otimes k}$$

Balanced polarization

If L is an ample vector bundle over a compact complex manifold X. Let $B = \{s_0, s_1, \cdots, s_{N_k}\}$ be a basis of $V_k = H^0(X, L^k)$ such that it gives an embedding

$$\Phi_k : X \longrightarrow \mathbb{P}(V^*), \qquad x \longrightarrow [s_0(x), \cdots, s_{N_k}(x)]$$

We choose the local frame e_L of L, then

$$s_\alpha(x) = f_\alpha(x) e_L^{\otimes k}$$

There is a $(N_k + 1) \times (N_k + 1)$ matrix

$$B_{\alpha\beta}(X) = \int_X \frac{f_\alpha \overline{f}_\beta}{\sum |f_\alpha|^2} dV$$

where

$$dV = \frac{(\Phi_k^*(\omega_{FS}))^n}{n! k^n}$$

Balanced polarization

Balanced polarization

Definition

The pair (X, L, B) is said to be balanced in $\mathbb{P}^{N_k}$ if

$$B(X) = \sigma \cdot I_{(N_k+1)\times(N_k+1)}$$

We also say that (X, L) can be balanced and $B = \{s_0, s_1, \cdots, s_N\}$ is a balanced basis.

It is obvious that

$$\sigma = \frac{Vol(X)}{N_k + 1}$$

Balanced polarization

Definition

The pair (X, L, B) is said to be balanced in $\mathbb{P}^{N_k}$ if

$$B(X) = \sigma \cdot I_{(N_k+1)\times(N_k+1)}$$

We also say that (X, L) can be balanced and $B = \{s_0, s_1, \cdots, s_N\}$ is a balanced basis.

It is obvious that

$$\sigma = \frac{Vol(X)}{N_k + 1}$$

If G is hermitian metric on the vector space $V_k = H^0(X, L^k)$, then we can choose an orthonormal basis $B = \{s_0, s_1, \cdots, s_{N_k}\}$ of V_k with respect to the metric G such that it gives the embedding

$$\Phi_k : X \longrightarrow \mathbb{P}(V^*), \quad x \longrightarrow [s_0(x), \cdots, s_{N_k}(x)]$$

Then there is an induced metric on L^k given by

$$FS(G) = \frac{1}{\sum G^{\alpha\beta} f_\alpha \overline{f}_\beta}$$

Balanced polarization

Balanced polarization

Definition

A hermitian metric G on $H^0(X, L^k)$ is called a balanced metric if the $\mathcal{L}^2$ metric on $H^0(X, L^k)$ induced by metric $FS(G)$ coincides with G, i.e.

$$(s_1, s_2)_G = \int_X \langle s_1(z), s_2(z) \rangle_{FS(G)} dV_z$$

for any $s_1, s_2 \in H^2(X, L^k)$.

Balanced polarization

Definition

A hermitian metric G on $H^0(X, L^k)$ is called a balanced metric if the $\mathcal{L}^2$ metric on $H^0(X, L^k)$ induced by metric $FS(G)$ coincides with G, i.e.

$$(s_1, s_2)_G = \int_X \langle s_1(z), s_2(z)\rangle_{FS(G)} dV_z$$

for any $s_1, s_2 \in H^2(X, L^k)$.

Theorem

If G is a balanced metric on $H^0(X, L^k)$ if

$$G_{\alpha\beta} = \frac{N_k + 1}{Vol(X)} \int_X \frac{f_\alpha \overline{f}_\beta}{\sum G^{\gamma\delta} f_\gamma \overline{f}_\delta} dV$$

with respect to the local basis $s_\alpha = f^\alpha e_L^{\otimes k}$

Balanced polarization

Balanced polarization

- There following arguments are equivalent:

Balanced polarization

- There following arguments are equivalent:

(1) There exists balanced metric on $H^0(X, L^k)$;

Balanced polarization

- There following arguments are equivalent:

(1) There exists balanced metric on $H^0(X, L^k)$;

(2) (X, L^k) can be balanced;

Balanced polarization

- There following arguments are equivalent:

(1) There exists balanced metric on $H^0(X, L^k)$;

(2) (X, L^k) can be balanced;

- Under the balance condition, the Begman kernel function is constant

$$B_k \equiv \frac{\dim_{\mathbb{C}} H^0(X, L^k)}{Vol(X)}$$

Constant scalar curvature and balance

Constant scalar curvature and balance

Theorem (Donaldson)

If (X, L^k) is balanced for every large k and ω_k is the balanced metric, and if ω_k converges to some limit ω_∞ in C^∞ as $k \longrightarrow \infty$, then ω_∞ has constant scalar curvature.

Constant scalar curvature and balance

Theorem (Donaldson)

If (X, L^k) is balanced for every large k and ω_k is the balanced metric, and if ω_k converges to some limit ω_∞ in C^∞ as $k \longrightarrow \infty$, then ω_∞ has constant scalar curvature.

Asymptotic Rieman-Roch expansion

$$B_k(\omega_k) = \frac{\dim_{\mathbb{C}} H^0(M, L^k)}{Vol(M, \omega_k)} = k^n + \frac{k^{n-1}}{2Vol(M, \omega_k)} \int_X \rho_{\omega_k} dV_k + O(k^{n-2}) \ (*)$$

Constant scalar curvature and balance

Theorem (Donaldson)

If (X, L^k) is balanced for every large k and ω_k is the balanced metric, and if ω_k converges to some limit ω_∞ in C^∞ as $k \longrightarrow \infty$, then ω_∞ has constant scalar curvature.

Asymptotic Rieman-Roch expansion

$$B_k(\omega_k) = \frac{\dim_{\mathbb{C}} H^0(M, L^k)}{Vol(M, \omega_k)} = k^n + \frac{k^{n-1}}{2Vol(M, \omega_k)} \int_X \rho_{\omega_k} dV_k + O(k^{n-2}) \quad (*)$$

Asymptotic expansion for the Bergman kernel

$$B_k(\omega_k) = k^n + \frac{1}{2}\rho_{\omega_k} k^{n-1} + \cdots$$

and

$$\|B_k(\omega_k) - k^n - \frac{1}{2}\rho_{\omega_k} k^{n-1}\|_{C^k} = O(k^{n-2}) \qquad (**)$$

Constant scalar curvature and balance

Theorem (Donaldson)

If (X, L^k) is balanced for every large k and ω_k is the balanced metric, and if ω_k converges to some limit ω_∞ in C^∞ as $k \longrightarrow \infty$, then ω_∞ has constant scalar curvature.

Asymptotic Rieman-Roch expansion

$$B_k(\omega_k) = \frac{\dim_{\mathbb{C}} H^0(M, L^k)}{Vol(M, \omega_k)} = k^n + \frac{k^{n-1}}{2Vol(M, \omega_k)} \int_X \rho_{\omega_k} dV_k + O(k^{n-2}) \quad (*)$$

Asymptotic expansion for the Bergman kernel

$$B_k(\omega_k) = k^n + \frac{1}{2}\rho_{\omega_k} k^{n-1} + \cdots$$

and

$$\|B_k(\omega_k) - k^n - \frac{1}{2}\rho_{\omega_k} k^{n-1}\|_{C^k} = O(k^{n-2}) \quad (**)$$

$$\Longrightarrow \quad \rho_\infty = \frac{1}{Vol(M, \omega_\infty)} \int_X \rho_{\omega_\infty} dV_\infty$$

Constant scalar curvature and balance

Constant scalar curvature and balance

Theorem (Donaldson)

Suppose $Aut(X, L)$ *is discrete and* ω_∞ *is a Kähler metric in the class* $2\pi c_1(L)$ *with constant scalar curvature. Then* (X, L^k) *is balanced for large enough* k *and the sequence of the balanced metrics* ω_k *converges in* C^∞ *to* ω_∞ *as* $k \to \infty$*.*

Constant scalar curvature and balance

Theorem (Donaldson)

Suppose $Aut(X, L)$ is discrete and ω_∞ is a Kähler metric in the class $2\pi c_1(L)$ with constant scalar curvature. Then (X, L^k) is balanced for large enough k and the sequence of the balanced metrics ω_k converges in C^∞ to ω_∞ as $k \longrightarrow \infty$.

Corollary

Suppose $Aut(X, L)$ is discrete. Then there is at most one Kähler metric of constant scalar curvature in the class $2\pi c_1(L)$.

Constant scalar curvature and stability

Constant scalar curvature and stability

Theorem (Zhang- Luo-Donaldson)

existence of cscK + discrete automorphism $\Longrightarrow$ *Balance*

Constant scalar curvature and stability

Theorem (Zhang- Luo-Donaldson)

existence of cscK + discrete automorphism $\Longrightarrow$ Balance

Chow-Mumford stability + Balance metrics converge $\Longrightarrow$ existence of cscK.

Constant scalar curvature and stability

Theorem (Zhang- Luo-Donaldson)

existence of cscK + discrete automorphism $\Longrightarrow$ *Balance*

Chow-Mumford stability + Balance metrics converge $\Longrightarrow$ *existence of cscK.*

Balance $\Longleftrightarrow$ *Chow-Mumford stability* $\Longrightarrow$ *Hilbert-Mumford stability.*